Partial functions and canonical extension

(Mai Gehrke and Brett McLean)

There are various operations one can perform on partial functions, such as composing two together. Thus algebraic structures can be obtained from collections of partial functions and selected operations [2]. This is exactly the same pattern as obtaining groups from collections of permutations. Algebras of partial functions are of interest to logicians, semigroup theorists, and computer scientists.

Canonical extension takes algebraic structures with an underlying partial order and 'completes' them, by embedding them into structures whose underlying order is a complete lattice [1]. Canonical extension can be extended to a functor.

The plan for this project is to first review both the concrete construction of canonical extensions that builds them from sets of prime filters, and the more abstract and general characterisation by algebraic properties. Then initiate a theory of canonical extension for algebras of partial functions, checking whether structures built from their prime filters have similar algebraic properties.

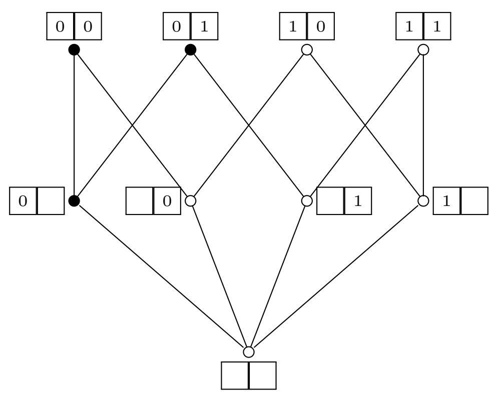

FIGURE 1. The partial functions on a two-element set (a prime filter marked)

Finally, a connection should be made with inverse semigroups, by extending the project to also encompass *injective* partial functions [3].

REFERENCES

[1] Mai Gehrke and John Harding, *Bounded lattice expansions*, Journal of Algebra **238** (2001), no. 1, 345–371.

[2] Marcel Jackson and Tim Stokes, *Modal restriction semigroups: towards an algebra of functions*, International Journal of Algebra and Computation **21** (2011), no. 7, 1053–1095.

[3] Mark V. Lawson, *A noncommutative generalization of stone duality*, Journal of the Australian Mathematical Society **88** (2010), no. 3, 385404.

Profinite integers, Stone duality and formal languages

(Mai Gehrke and Célia Borlido)

The ring of profinite integers $(\widehat{\mathbb{Z}}, +, \cdot)$ catches the attention of many researchers as it plays a role in different areas such as *infinite Galois theory*, *algebraic number theory* and *arithmetic geometry*. It admits several characterizations, amongst them as the *profinite completion* of the ring $(\mathbb{Z}, +, \cdot)$. Lenstra's paper [3] provides an elementary introduction to the subject, including some exercises.

On the other hand, in [2], Gehrke, Grigorieff and Pin interpreted *recognition* in *formal language theory* as an instance of Stone duality. In a further development of this work [1] it was proved that the *profinite completion of an algebra A* (of any type) is the *extended Stone dual* of the Boolean algebra (with some additional operations) of *recognizable subsets of A*.

The main goal of this project is to instantiate [2, 1] in the case of profinite integers. For this purpose, we would separately consider the Abelian group $(\widehat{\mathbb{Z}}, +)$ and the commutative monoid $(\widehat{\mathbb{Z}}, \cdot)$. It is well known that the languages recognized by $(\widehat{\mathbb{Z}}, +)$ over arbitrary finite alphabets are precisely those recognized by finite Abelian groups, and the first task would be to give a proof of this using duality. The next step would be to describe the languages recognized by the commutative monoid $(\widehat{\mathbb{Z}}, \cdot)$. Finally, the aim is to combine the results obtained for $(\widehat{\mathbb{Z}}, +)$ and for $(\widehat{\mathbb{Z}}, \cdot)$ and make a duality based study of the profinite ring of integers. In particular, one could explore its subrings and subgroups, its automorphisms, or in a slightly different direction, its algebraic elements.

References

[1] M. Gehrke. *Stone duality, topological algebra, and recognition.* J. Pure Appl. Algebra **220**(7):2711–2747, 2016.

[2] M. Gehrke, S. Grigorieff, and J.-É. Pin. *Duality and equational theory of regular languages.* In L. Aceto and al., editors, *ICALP 2008, Part II*, Vol. **5126** of Lect. Notes Comp. Sci., pages 246–257, Berlin, 2008. Springer.

[3] H. Lenstra. *Profinite number theory.* Eur. Math. Soc. Newsl. **100**:14–18, 2016.

Inverse semigroups and non-commutative Stone duality

(Clemens Berger and Célia Borlido)

Symmetry phenomena may be found everywhere, in areas ranging from chemistry, biology and physics to art and even in the literature. Mathematically, *symmetries* may be understood via the study of *groups*. However, the study of groups does not accommodate all interesting symmetry phenomena, as witnessed e.g. by *crystallography*, where one is in the presence of *partial symmetries*. The search of a suitable mathematical model to handle partial symmetries led, to the notion of *inverse semigroups* (a generalization of groups) independently by Wagner and Preston, and to the notion of *inductive groupoids* by Ehresmann. These two categories are in fact equivalent, due to the *Ehresmann-Schein-Nambooripad Theorem* (see e.g. [1]).

Inverse semigroups come naturally equipped with a partial order, and one of its strengths is that its set of idempotents forms a meet-semilattice. One may then consider the class of inverse monoids for which the partial order induces a semilattice structure on the whole semigroup and for which the set of idempotents admits a Boolean algebra structure. These assumptions together with a more technical one characterize what is called a *Boolean monoid*. In particular, every Boolean algebra is a (commutative) Boolean monoid. In [2], Lawson proved that the categories of Boolean monoids and of *Boolean groupoids* are dually equivalent. On the one hand, this work is a generalization of classical Stone duality, and on the other hand, it also contributes to a better understanding of the structure of inverse semigroups.

The first goal of this project is to study and understand the paper [2]. The prime example of an inverse semigroup is the set of partial injections on a set, equipped with (partial) composition. One may then explore what further structure is needed so that one obtains a Boolean monoid structure and the above mentioned theory applies.

References

[1] M. V. Lawson. *Inverse semigroups.* World Scientific Publishing Co., Inc., River Edge, NJ, 1998. The theory of partial symmetries.

[2] M. V. Lawson. *A noncommutative generalization of Stone duality.* J. Aust. Math. Soc. **88**(3):385–404, 2010.

Arithmetic partition complexes

(Frédéric Patras and Clemens Berger)

Dans un article plus cité en chimie qu'en mathématiques (et injustement méconnu), Peter Sellers a introduit dans [1] diverses familles de complexes et de groupes d'homologie qui apparaissent rétrospectivement liés à toute une variété de phénomènes : homologie des monoïdes et réécriture (une réaction chimique au sens de Sellers est assez proche d'une relation de réécriture, en un sens qui sera à explorer), notion de généricité en homologie, complexes de monômes et de partitions ... On doit par ailleurs à Louis Solomon [2] un article sur le même sujet, avec une approche un peu différente et complémentaire. Le but du mémoire sera d'étudier ces textes et de les relire (avec l'aide des encadrants) à la lumière de développements ultérieurs pour éventuellement en tirer un nouvel éclairage ou des compléments sur ces développements.

References

[1] P. Sellers. *Algebraic Complexes which Characterize Chemical Networks.* SIAM Journal on Applied Mathematics, Vol. **15**, No. 1 (Jan., 1967), pp. 13-68.

[2] L. Solomon. *The Complexes of Peter Sellers.* Adv. Math. **119**: 155-206, 1996.