Proposition de cours M2 pour le bloc Algèbre et géométrie

17 novembre 2019

1 1st term : Complex manifolds

Enseignant : S. Dumitrescu

Complex manifolds are important models and interesting geometric objects from the algebraic point of view and from the analytical point of view.
The aim of these lectures is to give an introduction to those different points of view and to the tools of different nature (analytical, algebraic, geometrical, topological...) used in the study of complex manifolds.
The course will adress the following main points :

- holomorphic functions of several variables
- Riemann surfaces, smooth complex manifolds, projective manifolds
- line bundles and divisors, holomorphic vector bundles
- differential forms of type (p, q), Dolbeault cohomology, de Rham cohomology
- sheaves and cohomology
- Riemann-Roch theorem on Riemann surfaces
- Abel-Jacobi theorem
- Kähler metrics
- harmonic forms and Hodge decomposition
- further developpements

An important focus will be on the study of Riemann surfaces and on the construction of various specific families of compact complex manifolds in higher dimension.
This first semester course is fitted to be attended in parallel with the lectures of S. Nivoche on functions of several complexe variables. It is also fitted to be an introduction to some geometrical aspects that will be deepened during the second semester in C. Simpson's course and in L. Stolovitch's course.

References :

1. Claire Voisin, Théorie de Hodge et géométrie algébrique complexe, SMF, Cours avancés, (2002).
2. Jean-Pierre Demailly, Complex analytic and algebraic geometry, book "OpenContentBook", available at http ://www-fourier.ujf-grenoble.fr/ demailly/books.html
3. Phillip Griffith and Joe Harris, Principles of algebraic geometry, Wiley, New York, (1978).
4. Jean-Benoît Bost, Introduction to compact Riemann surfaces, Jacobians and Abelians varieties, dans From number theory to physics, 64-211, Springer.

2 1st term : Introduction to Complex Analysis in several variables

Enseignante : S. Nivoche

The modern theory of functions of several complex variables can reasonably be dated to the researches of F. Hartogs and K. Oka in the first decade of the twentieth century.
The so-called Hartogs Phenomenon («Isolated singularity is removable, for any analytic function of several variables»), a fundamental feature that had eluded Weierstrass, reveals a dramatic difference between one-dimensional complex analysis and multidimensional complex analysis.

After important works in France (Cartan's seminar) and in Germany (H. Grauert and R. Remmert) quickly changed the picture of the theory. A number of issues were clarified, in particular that of analytic continuation. Here a major difference is evident from the one-variable theory : while for any open connected set D in C we can find a function that will nowhere continue analytically over the boundary, that cannot be said for $n > 1$. In fact the D of that kind are rather special in nature (a condition called pseudoconvexity).
Ever since the theory of functions of several complex variables has developed in various directions. One of these approaches is to use precise estimates on the Cauchy-Riemann equations in spaces of square integrable functions (a wide assortment of different L^2 norms with different plurisubharmonic functions) and differential forms to obtain quantitative understanding of holomorphic functions theory and analytic geometry on complex manifolds of higher dimensions. Referred to as the "L^2-theory of the dbar-problem", this method originated in the 1960s in the work of Hörmander, Kohn and Andreotti-Vesentini.
Since that time this theory has evolved into a very powerful and flexible tool to construct analytic objects.

Some aspects of the theory of holomorphic (complex analytic) functions, are essentially the same in all dimensions. The multidimensional theory reveals striking new phenomena (for power series expansions, integral representations, partial differential equations and geometry, for example).
In this course, we will introduce this theory and will sketch some of these phenomena.
There exists a large scope of the interaction between complex analysis and other parts of mathematics, including geometry, partial differential equations, probability, functional analysis, number theory, algebra, and mathematical physics.
One of the goals of this course will be to glimpse some connections between complex analysis and analytic geometry and functional analysis.
This first semester course is fitted to be attended in parallel with the lectures of S. Dumitrescu «Complex manifolds».

References :
S.-C. Chen and M.-C. Shaw,Partial Differential Equations in Several Complex Variables, AMS/IP Studies in Advanced Mathematics, vol. 19, 2001.
J. P. Demailly, Complex Analytic and Differential Geometry. Book OpenContentBook, available at http ://www-fourier.ujf-grenoble.fr/ demailly/books.html
J-P. Demailly, Analytic methods in algebraic geometry. Surveys of Modern Mathematics, International Press, Somerville, MA ; Higher Education Press, Beijing, 2012
H. Grauert and K. Fritzsche, Several Complex Variables, Springer-Verlag, New York, 1976.
R. C. Gunning, Introduction to Holomorphic Functions of Several Variables, 3 vols., Wadsworth and Brooks/Cole, Belmont, CA, 1990.
G. M. Henkin and J. Leiterer, Theory of Functions on Complex Manifolds, Birkhäuser, Boston, 1984.
L. Hörmander, An Introduction to Complex Analysis in Several Variables, Van Nostrand, Princeton, NJ,1966 ; 3rd ed., North-Holland, Amsterdam, 1990.
S. Krantz, Function Theory of Several Complex Variables, John Wiley and Sons, New York, 1982 ; 2nd ed.,Wadsworth, Belmont, CA, 1992.
R. Narasimhan, Several Complex Variables, University of Chicago Press, Chicago, 1971.
R. M. Range, Holomorphic Functions and Integral Representations in Several Complex Variables,Springer-Verlag, New York, 1986 ; 2nd. corrected printing, 1998.
B.V. Shabat, Introduction to complex analysis Part II. Functions of several variables (Translations of Mathematical Monographs). AMS. 1992.

Prerequisites : The following topics will be assumed known.
1. Basic theory of holomorphic functions of one complex variable.
2. Real Analysis : Basic facts about measure and integration in Euclidean spaces. Differential calculus of several real variables.
3. Basic facts about Banach spaces and Hilbert spaces.
4. Basic Algebra (vector spaces, groups, rings etc.).

3 1st term : Computational algebraic geometry

Enseignant : L. Busé

The classical results and open problems surrounding free resolutions, regularity and syzygies, topics that lie at the interface between commutative algebra and algebraic geometry, have undergone a striking evolution over the last quarter of a century, aided in large part by computer algebra calculations. Several new techniques have emerged and led to important theoretical developments with new results and new conjectures attracting of a lot of interest. In the same time, the applications of these techniques have been successfully applied in many fields such as combinatorics, geometric modeling, optimization, statistics, and it is now a very active area of research. The aim of this course is precisely to introduce students to some fundamental techniques and recent developments on effective methods in commutative algebra, with a view toward applications in computational algebraic geometry. Many examples will be treated and the students will be trained to the free computer algebra system Macaulay2 which is widely used and developed by a large community of mathematicians.

The first part of this course will be devoted to some of the main tools and concepts in commutative algebra that are used to derive effective methods : associated primes and primary decomposition, graded rings and modules, finite free resolutions, regular sequences, Hilbert functions and Gröbner basis. The second part of the lectures will focus on syzygies and their geometric content. We will develop material in relation with the Castelnuovo-Mumford regularity, a central and very active research topic, and modern elimination theory, with applications to the study of fibers of rational maps between projective spaces and the solving of polynomial systems. For that purpose, some classical material from homological algebra will be presented from a computational perspective, including Koszul complexes, Čech complexes, local cohomology, Tor and Ext functors, spectral sequences. Blowup algebras (symmetric and Rees algebras) will also be introduced and studied, in particular the shape and structure of their defining equations.

References :

1. David Eisenbud. Commutative Algebra with a view toward algebraic geometry. Graduate Texts in Mathematics, Vol. 150. Springer-Verlag, New York, 1995.
2. David Eisenbud. The Geometry of Syzygies : A Second Course in Algebraic Geometry and Commutative Algebra. Graduate Texts in Mathematics, Vol. 229, Springer, 2005
3. Hal Schenck. Computational Algebraic Geometry. Cambridge University Press, 2003.

4 2nd term : Symbolic reasoning and formal logics

Enseignant : C. Simpson

Subtitle : Machine Learning for the Classification of Algebraic Varieties
This course will be about the utilisation of Machine Learning for classification questions in mathematics, with a focus on the classification of algebraic varieties. The practical side will include learning the basics of programming deep neural networks with a standard package in Python. On the theoretical side we will discuss the classification landscape for algebraic varieties : varieties of general type, Calabi-Yau varieties, Fano and other rational varieties, and invariants such as Chern classes. We'll then put these together to program machine-learning on the classification question, motivated and guided by recent papers of Yang-Hui He on machine learning of the string landscape.

References :

1. He, Y. H. (2017). Machine-learning the string landscape. Physics Letters B, 774, 564-568
2. He, Y. H. (2017). Deep-learning the landscape. arXiv preprint arXiv :1706.02714
(and follow forward google citations).

5 2nd term : Local holomorphic dynamics and Cauchy-Riemann geometry

Enseignant : L. Stolovitch

Both in geometry and in dynamical systems, there are two main aspects of the (local) analysis of the objects : The first one is the behaviour at points where "nothing happens" : those points where a geometrical structure or a dynamical system are in an appropriate sense uniform.The other one is the behaviour at points where "things go wrong" : the geometry or the behaviour of the dynamical system changes in a drastic way. Such a point bears different names–depending on the field–called *singularity* or *catastrophe*. The aim of these lectures is to provide key concepts/tools for deciphering what is going on at such a bad point. We aim at defining a local model that is supposed to reflect the very nature of the considered object at the bad point. This *normal form* is obtained in a precise way by the action of a local group of transformations (fixing the bad point) on the original object. Transformation to Jordan normal form of matrices is a "baby" problem of that kind.
We shall mainly focus, on the one hand, on the study of isolated singularities of (germs of) holomorphic functions and on the other hand, on the study of (germs of) holomorphic dynamical systems (diffeomorphisms or vector fields) at a singular point (i.e a fixed point). These two unconnected problems can neverthless be treated in similar way. For dynamical systems, we shall define conditions that allows to consider the dynamcial system obtained by "linearizing at the singular point" as good holomorphic model. This requires to develop the concepts of *resonances* and *small divisors*. We shall show the existence of an infinite number of resonances is a serious obstruction to obtaining a good holomorphic model. This can still be achieved on very rare situtations, named *completely integrable*. Finally, we shall show how dynamics can infer deep understanding of the geometry of some real submanifolds in the complex Euclidean space, those which have a singularity of its Cauchy-Riemann structure. This course relies on S. Dumitrescu's and S. Nivoche's courses.

References :

1. Arnold, V.I. Geometrical methods in the theory of ordinary differential equations., Springer-Verlag, New York, 1988.

2. Arnold, V.I. ; Gusein-Zade, S. M. ; Varchenko, A. N. Singularities of differentiable maps. Vol. I. The classification of critical points, caustics and wave fronts. Birkhäuser, 1985.

3. Bruno, A.D. Analytical form of differential equations, Trans. Moscow Math. Soc. 25 (1971), 131–288, 26 (1972), 199–239.

4. Moser, J. and Webster, S. M., Normal forms for real surfaces in $\mathbf{C}^2$ near complex tangents and hyperbolic surface transformations, Acta Math., 150, 1983, 3-4, 255–296.

5. Stolovitch, L. Normal form of holomorphic dynamical systems. Hamiltonian dynamical systems and applications, 249–284, NATO Sci. Peace Secur. Ser. B Phys. Biophys., Springer 2008.